Games, graphs, and machines

Partial orders

August 6, 2024

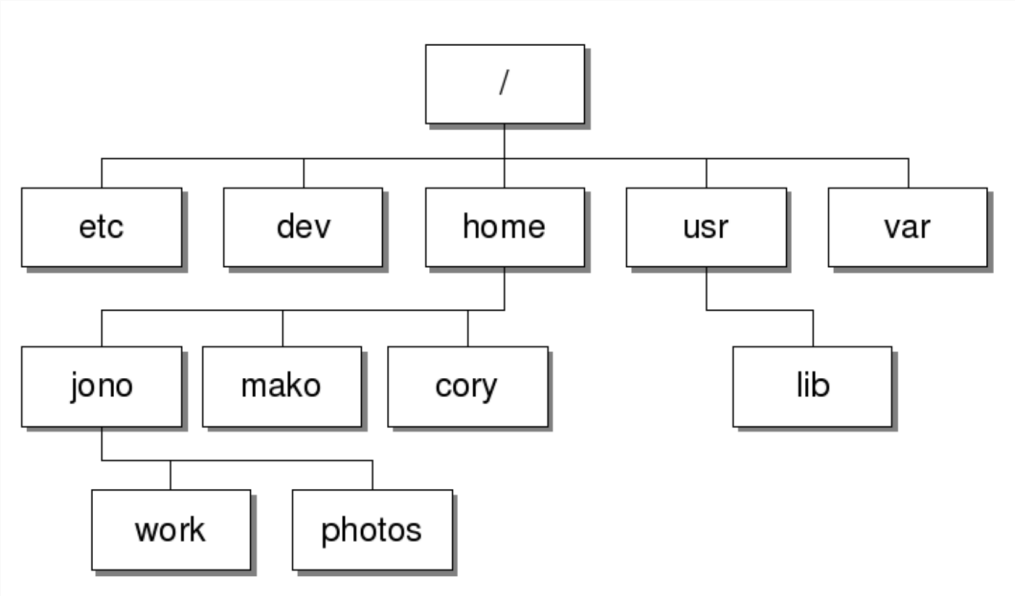
A partial order is

1. Reflexive
2. Anti-symmetric
3. Transitive

2. For $a \neq b$, cannot have

$a \to b$ and $b \to a$

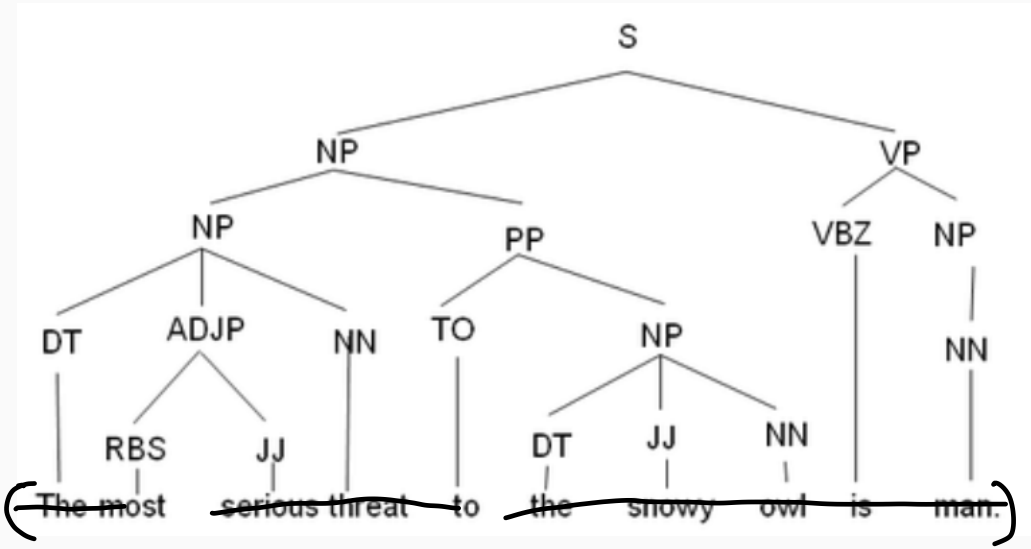
Partial orders from real life



S = Directories on a computer

$a \leq b$ if a is contained in b

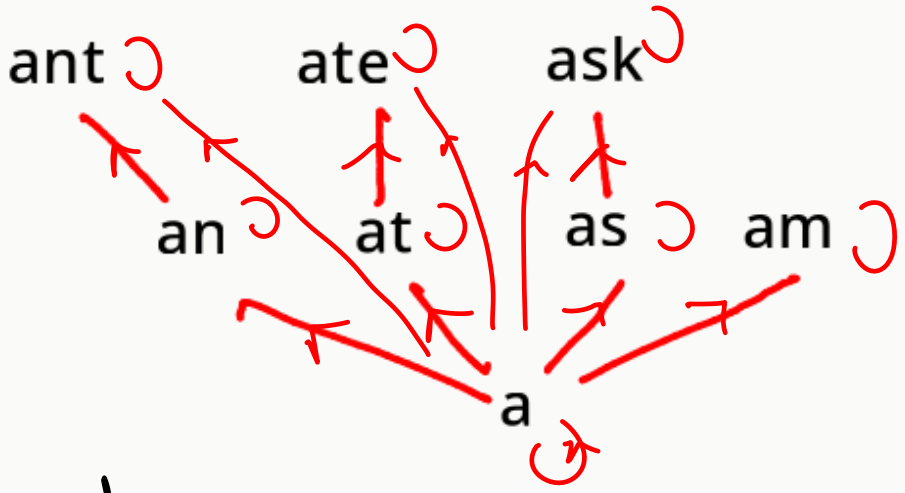
Partial orders from real life



S = Sentence fragments

$a \leq b$ if a frag is contained in b frag

Partial orders from real life



S = words

$u \preceq v$ if v starts with u

u is a prefix of v.

prefix

Partial orders from real life

S = Subdivisons of 

$a \leq b$ if a is a further subdiv of b.

Paper, Scissors, Rock: Partial order?

- Paper $\preceq$ Scissors
- Scissors $\preceq$ Rock
- Rock $\preceq$ Paper

$P \preceq P$

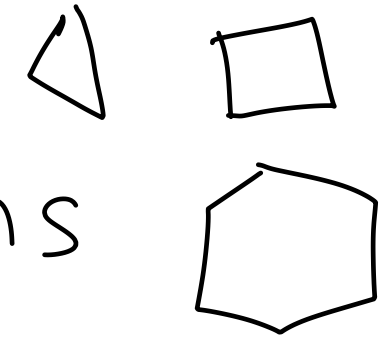
$R \preceq R$

$S \preceq S$

Not transitive!

Cannot make this anti sym & transitive.

S = Set of all planar polygons



$P \leq Q$ if $\text{area}(P) \leq \text{area}(Q)$.

A total order is a partial order where any two are comparable.

(1) Partial order ?

(2) Total order ?

(3) Not ✓ Not antisymmetric

Same area

$\leq$ $\geq$

Divisor poset

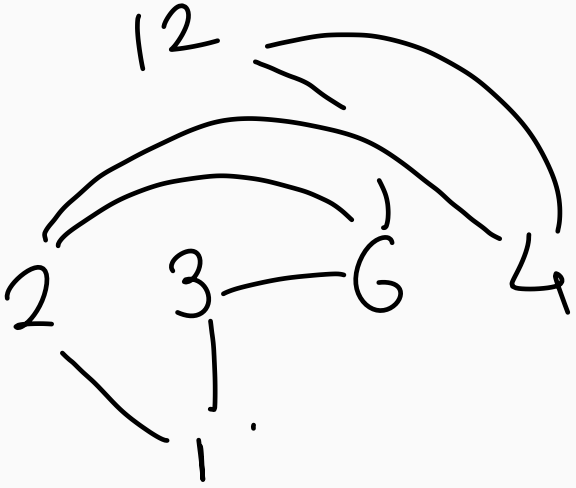
Let $S = \{1, 2, 3, 4, 6, 12\}$.

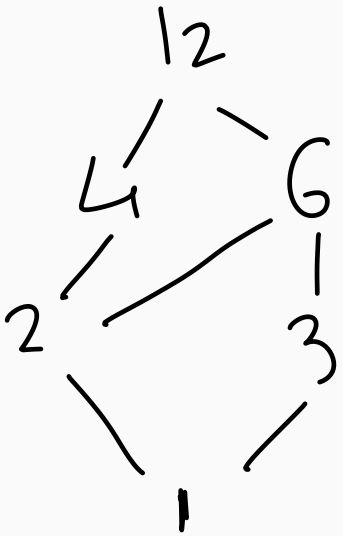
Say $a \preceq b$ if a divides b.

What is the Hasse diagram?

Divisor poset of 12.

"Partially ordered set"





Subset poset

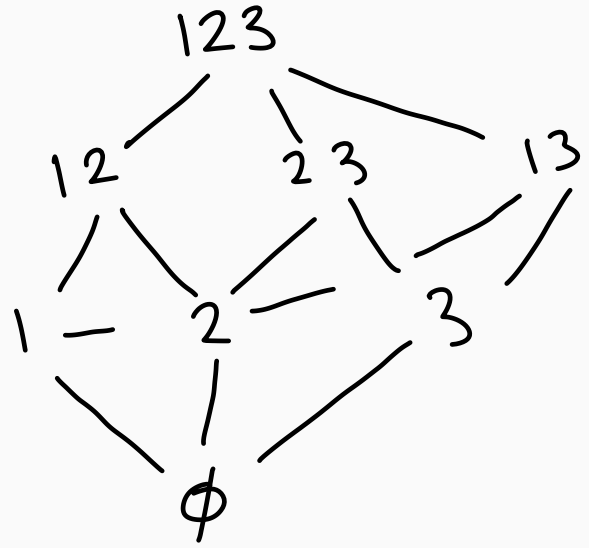
Let $S = \text{Pow}(\{1, 2, 3\})$.

Say $A \preceq B$ if $A \subset B$. $= A \subseteq B$ } Subset poset of $\{1,2,3\}$.

What is the Hasse diagram?

ϕ . $\{1\}$. $\{2\}$. $\{3\}$. $\{1,2\}$ $\{1,3\}$ $\{2,3\}$. $\{1,2,3\}$

12 13 23



Product poset

Let $\leq$ be the usual order on $\mathbb{R}$. Define $\preceq$ on $\mathbb{R} \times \mathbb{R}$ by

$$(a, b) \preceq (c, d) \text{ if } a \leq b \text{ and } c \leq d.$$

1. Give an example of two incomparable elements under $\preceq$.
2. Plot all elements that are $\preceq (2, 3)$.
3. Plot all elements (x, y) with $(1, 1) \preceq (x, y) \preceq (2, 3)$.

Tomorrow

Max/min

In all the examples so far, identify

- the maximum (if it exists)
- the minimum (if it exists)
- all maximal elements
- all minimal elements

Tomorrow